

# Design of variable densities for least-squares approximations

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**Abstract**—We study the problem of interpolating a signal using samples at coordinates drawn for a probability density over the domain of definition of the signal, with the assumption that it can be approximated in a known linear subspace. Our goal is to minimize the number of samples needed to ensure a well-conditioned estimation of the signal. We show that the problem of optimizing the probability density is convex, and that applying the Frank-Wolf algorithm yields a simple and interpretable optimization procedure. Examples of optimizations are given with polynomials, trigonometric polynomials and Fourier-Bessel functions for wavefield interpolation.

## I. INTRODUCTION

The problem of sampling, or interpolating, a signal or a function is fundamental in signal processing. Classical examples include sampling of band-limited signals, with the Nyquist-Shannon theorem ensuring that a signal with limited bandwidth can be perfectly recovered using uniformly spaced samples of sufficient density, and polynomial approximation, where stability requires non-regularly spaced samples (the instability of polynomial interpolation with equispaced samples is known as the Runge Phenomenon).

We consider here that the sampling points are chosen according to a probability density  $\mu$  over the domain of definition of the signal. Our goal is then to optimize the density  $\mu$ , to ensure a stable interpolation with a number of samples as low as possible.

Recently, in the domain of compressed sampling, it has been shown that non-uniform sampling densities, so-called variable densities [1], [2], can either improve the reconstruction quality with constant number of samples, or reduce the number of samples for a given quality.

We consider here a simpler linear model, investigated in [3] where the signal  $f$  to recover can be approximated in a sequence of linear subspaces  $V_m$  of increasing dimension  $m$ , i.e. that the error between the best approximation  $f_m$  in  $V_m$ , defined by

$$f_m = \operatorname{argmin}_{\tilde{f} \in V_m} \|f - \tilde{f}\|^2,$$

and  $f$  tends to 0. The choice of these spaces (e.g. linear combinations of (trigonometric) polynomials, Fourier-Bessel functions, etc.) depends on the properties of the signal to be recovered, and the order  $m$  is chosen to ensure a small approximation error.

For a given space  $V_m$  (note that the choice of  $V_m$  is out of scope of this paper), we use a simple least-squares estimator

using samples of the signal at coordinates  $x_i$  to build an estimate  $\tilde{f}_m$  of  $f$ :

$$\tilde{f}_m = \operatorname{argmin}_{\tilde{f} \in V_m} \sum_{i=1}^N |f(x_i) - \tilde{f}(x_i)|^2.$$

The  $N$  sampling points  $x_i$  are drawn using the sampling density  $\mu$ . The objective is here to optimize  $\mu$  such that a minimal number of measurements is needed to ensure stability of the estimation, i.e. that the estimation error is of the same order as the approximation error:

$$\|\tilde{f}_m - f\| \leq C \|f_m - f\|$$

where  $C$  is a constant sufficiently close to 1. We will show that the criterion introduced in [3], related to the number of sampling points necessary to ensure stability, is actually convex with respect to the density  $\mu$ . Examples of optimized densities are given for selected linear models, using the Frank-Wolfe algorithm, and simulations show their superiority compared to simpler densities. In the examples given here, optimized densities have the interesting property of being sparse, i.e. are concentrated on a low number of points, equal or close to the dimension  $m$ .

### A. Previous works

A convex method for sensors selection, based on a convex relaxation of a non-convex problem, was proposed in [4]. A related method, based on frame theory, is given in [5]. Both approaches are related to the problem studied here, but they are mainly concerned with parameter estimation, instead of signal interpolation. In particular, choosing a different family spanning the same space will modify the sensors selected by these methods, while this has no influence when interpolating a signal.

A related and more complicated problem is the design of variable densities for compressed sensing [1], [2]. Compressed sensing was initially developed using samples drawn from uniform densities, but it has been shown that using a non-uniform density can improve reconstruction with a constant number of samples, or reduce the number of measurements for a given quality. We consider here a simpler problem, where the model is linear, allowing simpler analysis and algorithms.

Finally, the  $K(m)$  criterion we use here (see eq. (1)) was used to design microphone arrays in [6] and [7]. However,

$K(m)$  was not optimized directly, but used to compare sampling densities designed using heuristics. We here directly optimize the sampling density  $\mu$  to minimize  $K(m)$ .

### B. Structure of the paper

Least-squares approximations and the  $K(m)$  criterion are recalled in section 2. The optimization method is introduced in section 3, and numerically tested in section 4. Section 5 concludes the paper.

## II. LEAST-SQUARES APPROXIMATIONS

In [3], a criterion for the stability of least-squares approximations is proposed. An approximation to a function in a space  $V_m$  is obtained by a finite dimensional least-squares problem using samples drawn from a probability distribution  $\mu$ . The space  $V_m$  is equipped with a basis  $q_j$ , orthogonal with respect to the scalar product defined by  $\mu$ . The quantity  $K(m, \mu)$  is defined by

$$K(m, \mu) = \sup_x \sum_{j=1}^m |q_j(x)|^2. \quad (1)$$

$K(m, \mu)$  is increasing with  $m$ , is bounded from below by  $m$ , and depends on  $\mu$  through the orthogonal basis  $q_j$ . The main result in [3], is that if the number of measurements  $N$  satisfies

$$K(m, \mu) \leq \kappa \frac{N}{\log N}$$

for a given constant  $\kappa$ , then the expectation of the estimation error  $\|\tilde{f}_m - f\|$  is of the same order as the approximation error  $\|f_m - f\|$ , measured with the density  $\mu$ .  $K(m, \mu)$  essentially measures the number of samples necessary to ensure a good reconstruction of the function  $f$ .

This result suggests that a good sampling density  $\mu$ , in the sense that a minimal number of samples is required, is such that  $K(m, \mu)$  is equal or close to  $m$ . In [3], no method was given to design a sampling density with a low  $K(m, \mu)$ .

In this paper, we tackle this problem in the discrete case, i.e. where a large vector of dimension  $L$  is approximated in a subspace of lower dimension  $V_m$ . The density  $\mu$  is replaced by a vector of weights  $w_i$  of dimension  $L$ .

## III. OPTIMIZATION METHOD

To design a sampling density for the least-square approximation problem, we aim at solving the optimization problem

$$\hat{w} = \operatorname{argmin}_w K(m, w) \text{ subject to } \sum_{i=1}^L w_i = 1, w_i \geq 0. \quad (2)$$

The constraints on  $w$  are convex (they define the probabilistic simplex  $\Delta$ ), and as will be shown in the next subsection,  $K(m, w)$  is convex with respect to  $w$ .

We denote  $Q^w$  the matrix of an orthogonal basis of  $V_m$  with respect to  $w$ , i.e. its column vectors are orthogonal:

$$\sum_{i=1}^L w_i q_{ik}^* q_{ik} = \delta_{kj}.$$

We call  $q_k^w$  the row vectors of  $Q^w$ .

### A. Convexity

For a given probability density  $w$ ,  $K(m, w)$  is the largest of the  $K_i(m, w)$  defined by

$$K_i(m, w) = \sum_{j=1}^m |q_{ij}^w|^2 = |\langle q_i^w, q_i^w \rangle|^2.$$

It is therefore sufficient to prove that the  $K_i$  are convex.

For a different density  $w'$ , an orthogonal basis can be found using the Gramian matrix  $G$ , defined by

$$G_{ij} = \sum_{k=1}^L w'_k q_{ki}^* q_{kj}.$$

Using the expression of the new basis

$$Q^{w'} = Q^w G^{-1/2},$$

it is shown that the derivative of  $K_i$  with respect to  $w_k$  is

$$\frac{\partial K_i}{\partial w_k} = -|\langle q_i^w, q_k^w \rangle|^2. \quad (3)$$

Similar computations and the application of the Schur product theorem show that the Hessian matrix of  $K_i$  is symmetric positive. The  $K_i$  being convex, the quantity  $K(m, w)$  is thus a convex function of the sampling density.

### B. Optimization algorithm

A large number of optimization algorithms are available to solve the convex problem (2) (projected gradient or subgradient descents, interior point methods, etc.). We choose here to use the Frank-Wolfe algorithm (or conditional gradient) [8], [9]. This choice is justified by the following arguments:

- no projection on the feasible set is needed
- iterates are likely to be sparse
- a very simple interpretation of the algorithm is available.

Given an iterate  $w_k$ , the Frank-Wolfe algorithm consists of the following steps:

- find  $s_k$  minimizing the linear approximation of the objective function
- fix a step size  $\gamma = 2/(k+2)$
- set the next iterate as a convex combination of the current iterate and the minimizer  $s_k : w_{k+1} = w_k + \gamma(s_k - w_k)$ .

As  $s_k$  and  $w_k$  are both in  $\Delta$ ,  $w_{k+1}$  is also in  $\Delta$  and no projection is needed.

In our case, the gradient of  $K$  is the gradient of the maximum  $K_i$ , with index  $i^*$ . The linear problem to solve is

$$\min \langle s, \operatorname{grad} K_{i^*} \rangle \text{ subject to } \sum_{i=1}^L s_i = 1, s_i \geq 0.$$

Its solution is a vector with zero coefficients, except at the minimal value of the gradient, where it is one. A straightforward application of the Cauchy-Schwartz inequality to eq. (3) also shows that the minimal coordinate of the gradient of  $K_i$  is the  $i$ -th coordinate.

An iteration of the Frank-Wolfe algorithm applied to our optimization problem has thus the simple implementation

- find the index  $i^*$  of the maximum value of  $K_i$ ,
- add some weight on the  $i^*$ -th coefficient of the sampling density,
- rescale the sampling density such that  $\sum_{i=1}^L w_i = 1$ .

The cost of the iteration is dominated by the orthogonalization of the basis for the space  $V_m$  with respect to the weights  $w_i$ , needed to compute  $K_i$ .

#### IV. NUMERICAL RESULTS

The optimization method is now applied to three cases of least-squares approximation: polynomial approximation, trigonometric polynomials, and Fourier-Bessel functions for wavefields. In the three cases, values close to  $m$  can be reached.

The density is optimized using 1000 iterations of the Frank-Wolfe algorithm. For the polynomial and trigonometric polynomial cases, the error of interpolation for the Runge function, defined on  $[-1, 1]$  by  $r(t) = 1/(1 + 25t^2)$  is estimated from 1000 different sampling sets for a varying number of points.

##### A. Polynomials

In [3], it was shown that interpolation using sampling points drawn from a uniform density is unstable unless a number of points scaling like the square of the order is used, and that using the density

$$\mu_0 = \frac{\pi}{2\sqrt{1-t^2}}$$

for polynomials defined on the interval  $[-1, 1]$  yields a value of  $K$  linear with  $m$ , but larger:  $K_0(m) = 2m - 1$ .

Figure 1 shows the density obtained after 1000 Frank-Wolfe iterations (computation time of approx. 0.4s) for order 15 approximation (i.e.  $m = 16$ ). The value of  $K(m)$  during the optimization is plotted on figure 2. The final density has  $K_{opt}(16) = 16.11$ . It is concentrated on 16 clusters, with coordinates close the Chebyshev nodes indicated by circles. In comparison to the continuous density  $\mu_0$  (with  $K_0(16) = 31$ ), this optimized density is dependent of  $m$ .

The interpolation error with samples drawn from the uniform and the optimized densities is plotted on Fig. 3. When few measurements are available, better results are obtained with the optimized density, while the uniform density performs better when more measurements are available (this to be expected as uniform sampling needs more samples, but ensure stability in the unweighted  $\ell_2$  norm).

For low numbers of measurements, it is unlikely that all 16 clusters contain at least a sampling point. In that case, the linear system to be inverted is ill-conditioned. On Fig. 1, the interpolation error is not plotted if ill-conditioning occurs for at least one realization of the sampling set.

To avoid this ill-conditioning, it is possible to use one point in each cluster. In this case, the sampling set contains 16 points, as many as the dimension of  $V_m$ . The error is slightly larger, but comparable, to the error when using Chebyshev nodes, and smaller than the error for equispaced points. Interpolation errors for these deterministic choices are indicated by circles on Fig. 3.

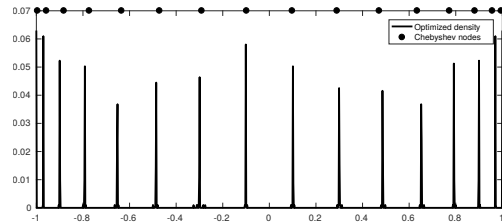


Fig. 1. Optimized density for polynomials of order 9 and Chebyshev points.

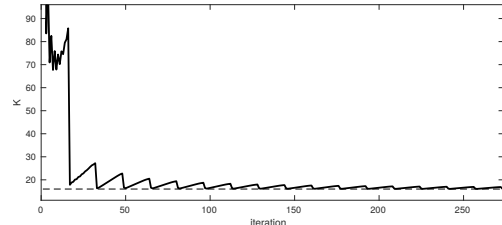


Fig. 2. Values of  $K$  during the optimization. The horizontal lines indicates the lower bound.

##### B. Trigonometric polynomials

For standard trigonometric polynomials  $e^{int}$ ,  $-L \leq n \leq L$  on  $[-\pi, \pi]$ , the uniform density is trivially optimal. In the case where the interval is reduced to  $t \in [-\alpha\pi, \alpha\pi]$ ,  $\alpha < 1$ , the uniform density is no longer optimal. Fig. 4 shows the density optimized using the proposed method for  $L = 8$ , i.e.  $m = 17$ . Like in the polynomial case, the density is concentrated on as many clusters as the dimension of the approximation space.

Estimation errors for uniform density, optimized density, Chebyshev nodes and points located on the clusters are plotted on Fig. 5. Here, choosing the clusters of the optimized density yields better results than Chebyshev nodes.

##### C. Fourier-Bessel

The final example is inspired by [6] and [7], for the design of microphone arrays. For simplicity, only the 2D case is considered here. A soundfield  $p$  in harmonic regime, solution to the Helmholtz equation, is approximated on a convex domain by finite sums of Fourier-Bessel functions

$$p \approx \sum_{l=-L}^L a_l e^{il\theta} J_l(kr)$$

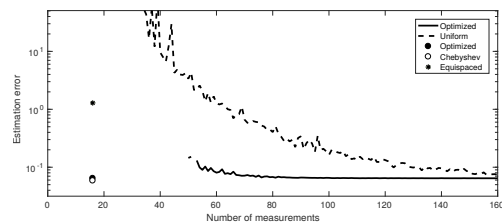


Fig. 3. Interpolation error with polynomials of order 9, for samples drawn from the uniform and optimized densities. The circles indicates interpolation error for deterministic choices of points : Chebyshev nodes, nodes at the clusters of the optimized density, and equispaced nodes.

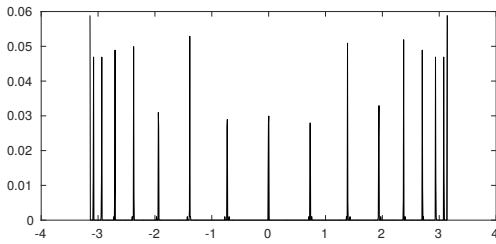


Fig. 4. Optimized density for trigonometric polynomials of order 8,  $m = 17$  for  $\alpha = 0.7$ .

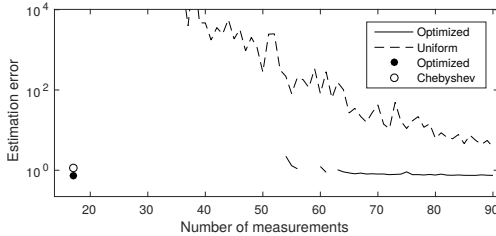


Fig. 5. Interpolation error with trigonometric polynomials,  $L = 8$ ,  $m = 17$ , for uniform and optimized density. Interpolation error for deterministic nodes (Chebyshev nodes, and nodes at the clusters of the optimized density) are indicated by circles.

in polar coordinates, where  $J_l$  is the Bessel function of order  $l$ .

It was shown for particular cases of domains that densities  $w$  that have a large proportion of measurements on the border of the domain have a  $K(m, w)$  that scales linearly with  $m$ . We here apply the method to a family of  $m = 21$  Fourier-Bessel functions ( $L = 10$ ) defined on a square. The obtained density is pictured on Fig. 6, with  $K(21, w) = 21.25$ . In accordance with the previous results, most of the density is concentrated on the border of the domain, with only 9% in the interior of the domain. Similarly to the previous cases, the density is concentrated on a few clusters. However, in this case, they are more clusters (28) than the dimension of the spaced spanned by the Fourier-Bessel functions (21).

Interpolation error is plotted on figure 7. In this example, measurements are corrupted by a white gaussian noise, with SNR = 8 dB. The errors using a regular grid of 25 points, and 21 points chosen from the support of the optimized density (one inside, 20 on the border) are also indicated. As in the polynomial case, interpolation for low number of samples and the optimized density is likely to be unstable, explaining the high interpolation errors for less than 50 samples.

## V. CONCLUSION

A simple method to optimize sampling densities is presented, in the form of a convex optimization problem. On the examples shown in this paper, the algorithm can reach the lower bound for  $K(m, w)$ . The extension to the continuous case, as well as the proof that  $K(m, w) = m$  can always be reached, is the subject of current research.

The numerical results (in particular in the polynomial case) suggest that the optimal density actually depends on the num-



Fig. 6. Optimized density for 21 Fourier-Bessel functions on a square.

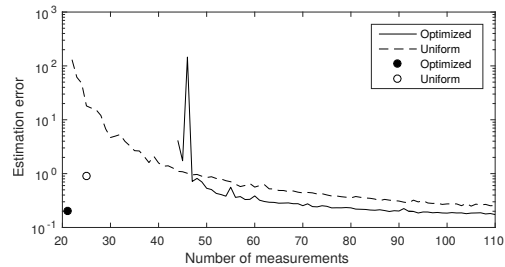


Fig. 7. Interpolation error of an acoustical source with uniform and optimized densities, with  $L = 10$ ,  $m = 21$ . Points indicate a regular grid of 25 points, and a non-uniform grid of 1 points chosen from the support of the optimized density.

ber of available measurements. As the optimized density are sparse, it is possible to directly select the sampling points from the optimized density. Such a selection yields better results than simple independent draws, with number of measurements close or equal to the dimension of the finite dimensional approximation space.

The method is limited to approximation in linear spaces. In particular, it cannot deal with optimization of variable densities in compressed sensing. Adaptations of the framework presented here to non-linear, sparse models will be investigated.

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