

A BLOCK-SPARSE MUSIC ALGORITHM FOR THE LOCALIZATION AND THE IDENTIFICATION OF DIRECTIVE SOURCES

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ABSTRACT

We introduce a generalization of the MUSIC algorithm to treat block-sparse signals in a multi-measurement vector framework. We show, through theoretical analysis and numerical experiments, that the requirements in terms of number of snapshots and number of measurements depend not only on the sparsity and on the size of the blocks, but also on the rank of the matrices of coefficients for each block. We apply this algorithm to the localization of directive sources, which can be modeled by block-sparsity in a dictionary of multipoles, and show that it compares favorably to a greedy approach based on the same model.

Index Terms— source localization, block sparsity, multi-measurement vector, multiple signal classification

1. INTRODUCTION

The MUSIC algorithm has recently raised interest in the sparsity and compressed sensing community, for its use as a multi-measurement vector (MMV) sparse recovery algorithm, where several vectors are partially measured, and reconstructed with the knowledge that they share the same support [1, 2]. Indeed, in contrast to algorithms such as Simultaneous Orthogonal Matching Pursuit (SOMP) [3], it is capable of taking in account the diversity of the signals in the different channels. We introduce here a generalization of MUSIC to the case of MMV block-sparsity, where the coefficients of the signals are grouped in blocks, and only a few of these blocks are involved in the decomposition.

The original motivation of the development of the MUSIC algorithm was the localization of far-field sources [4]. It is assumed that the sources are located at a large distance from the sensor array, making the directivity of the sources negligible. We show here that our MUSIC generalization can deal with directive sources in near-field, and that the particular model used in this case is more constrained than a standard MMV block-sparse model.

Prior work on source localization and sparse recovery is discussed in section 2. We introduce the generic block-sparse model in section 3, as well as two particular examples. The block-sparse MUSIC algorithm is described in section 4. A theoretical analysis of the algorithm is given in section 5. Numerical results with random dictionaries and source localization are given in section 6, and concluding remarks are given in section 7.

2. PRIOR WORK

The MUSIC algorithm [4] was proposed by Schmidt to locate far-field sources. The field radiated by S sources is measured at an array of sensors at multiple times. The vector of measurements at time t_l is given by

$$\mathbf{m}_l = \sum_{k=1}^S \alpha_{kl} \mathbf{a}(\theta_k)$$

where \mathbf{a} is the array manifold, deduced from the geometry of the array, θ_k is the direction of arrival of the k -th source, and α_{kl} its complex amplitude at time t_l .

Assuming that the sources are uncorrelated and the number of sensors is sufficient, the matrix \mathbf{M} , obtained by concatenating the vectors \mathbf{m}_l , has rank S . To identify the directions of arrival, we compute the pseudo-spectrum

$$p(\theta) = \frac{1}{\|P_{\mathbf{M}}^{\perp} \mathbf{a}(\theta)\|^2} \quad (1)$$

where $P_{\mathbf{M}}^{\perp}$ is the projection on the orthogonal of the space spanned by the columns of \mathbf{M} (the noise subspace). If a source is present at direction θ , then $\mathbf{a}(\theta)$ is included in the space spanned by \mathbf{M} (the signal subspace) and $p(\theta) = \infty$. With noisy measurements, the signal subspace is estimated as the space spanned by the first S left singular vectors of \mathbf{M} .

A variant of MUSIC was recently proposed for the treatment of directive sources [5]. Based on the phase of the signals only, it was shown to allow the localization of pure monopoles, dipoles and quadrupoles. However, it is not clear if the method can localize more general sources, such as combinations of such elementary sources, and estimate their directivities.

Source localization and the MUSIC algorithm are closely connected to sparse representations. As the source localization problem is a special case of sparse recovery [6], methods based on sparsity can be used, for the standard source localization problem, or for directive sources localization, using a block-sparse modeling of the source [7]. Conversely, the MUSIC algorithm can also be used as a sparse recovery algorithm for the MMV problem, like shown in [1], and can even be combined with OMP [2]. We propose an extension of this application of MUSIC to block sparsity.

3. SIGNAL MODEL

We introduce here the MMV block sparse model that we consider in this work. In the standard MMV model, the goal is to reconstruct a matrix \mathbf{X} from measurements \mathbf{Y} given by $\mathbf{Y} = \mathbf{D}\mathbf{X}$ where \mathbf{D} is a known matrix called the dictionary, with the knowledge that only

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a few rows of \mathbf{X} are non-zero. This corresponds to a set of sparse signals \mathbf{x}_i (the columns of \mathbf{X}) that share the same support. This is in particular the case for source localization using multiple observations (snapshots) of the sources, assuming that the sources do not move between the observations.

For the MMV block-sparse model, a particular structure is imposed on \mathbf{D} and \mathbf{X} : the dictionary \mathbf{D} is assumed to be composed of L sub-dictionaries \mathbf{D}_j of size $N_m \times N_c$, and the matrix \mathbf{X} is the vertical concatenation of L sub-matrices \mathbf{X}_j , of size $N_c \times N_s$:

$$\mathbf{Y} = \begin{pmatrix} \mathbf{D}_1 & \cdots & \mathbf{D}_L \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_L \end{pmatrix}$$

N_m is the number of measurements, N_c the number of components in a block, and N_s the number of observations.

Each sub-matrix \mathbf{X}_j contains the coefficients of the atoms of the sub-dictionary \mathbf{D}_j for the N_s observations. We assume that only a few of these sub-matrices \mathbf{X}_j are nonzero, and call the number of such matrices the sparsity S of the matrix \mathbf{X} .

The standard sparsity model is obtained when $N_c = N_s = 1$, the mono-channel block-sparse model when $N_s = 1$, and the MMV model when $N_c = 1$. In our case, a further parameter is the rank R of the nonzero sub-matrices \mathbf{X}_j , that can vary between 1 and $\min(N_c, N_s)$. This rank is obviously 1 for the previously investigated models, and was not taken into account. We will show that this rank is involved in the minimal numbers of measurements and snapshots needed to guarantee perfect reconstruction.

Spectral analysis is a simple application of this model. Given a signal

$$x_n = \sum_{l=1}^S a_l \cos(\omega_l n + \phi_l) = \sum_{l=1}^S a_l e^{i\phi_l} e^{i\omega_l n} + a_l e^{-i\phi_l} e^{-i\omega_l n},$$

the goal is to estimate the frequencies ω_l (assumed to be nonzero). MUSIC can be applied on the matrix $\mathbf{Y} = (x_{n+m})_{nm}$, which is $2S$ sparse in the dictionary of complex exponentials \mathbf{D} . By grouping the conjugate sinusoids, we obtain a block sparse model, with blocks of size 2. The rows of a nonzero sub-matrix \mathbf{X}_j are $(a_j e^{i\phi_j} e^{i\omega_j m})_m$ and $(a_j e^{-i\phi_j} e^{-i\omega_j m})_m$, which are non-colinear when $\omega \neq 0$, thus having full rank.

We now turn to an example where the rank of the sub-matrices \mathbf{X}_j is 1, localization of sources with non-isotropic directivity patterns. We will assume that the directivities of the sources are limited to combinations of monopoles and dipoles. The field emitted by a source located at the origin is, in 3D

$$p(\vec{r}) = a \left(\alpha h_0(kr) + \sum_{l=-1}^1 \beta_l Y_{1l}(\vec{r}/r) h_1(kr) \right).$$

where Y_{1m} are the first order spherical harmonics, h_0 and h_1 spherical Hankel functions, a is the complex amplitude of the source, and α and β_l characterize the directivity of the source. In far-field localization, we assume that the sources do not move between the observations. Here a further assumption is that their directivity patterns relative to the sensor array remain constant (while it is a mild assumption to assume that their directivity with respect to a local frame is constant, we also need that the sources do not change their orientation with respect to the sensor array).

In this case, the measurements for a source at position \vec{x}_0 at sensors \vec{y}_n for the N_s snapshots can be written

$$m_{ns} = a_s \left(\alpha y_0(kr_n) + \sum_{m=-1}^1 \beta_l Y_{1l}(\vec{r}_n/r_n) h_1(kr_n) \right)$$

where $r_n = \vec{x}_0 - \vec{y}_n$. The corresponding sub-matrix \mathbf{X}_s is

$$\mathbf{X}_s = \begin{pmatrix} a_1 \alpha & \cdots & a_{N_s} \alpha \\ a_1 \beta_{-1} & \cdots & a_{N_s} \beta_{-1} \\ a_1 \beta_0 & \cdots & a_{N_s} \beta_0 \\ a_1 \beta_1 & \cdots & a_{N_s} \beta_1 \end{pmatrix}$$

We see here that not only a few matrices are nonzero, but also that these matrices are of rank 1. As shown in the next section, this further constraint on the sparse model will allow to use less measurements and snapshots to locate and characterize the sources.

4. ALGORITHM

A simple way to treat block sparsity is to first forget the structure of the dictionary (i.e. treating the problem as a standard MMV problem), and apply the MUSIC algorithm. The blocks can then be estimated a posteriori using the identified atoms. While this method would work when the rank of the coefficient matrices \mathbf{X}_j is full, it will likely fail if not. Indeed, in this case, nothing guarantees that at least one of the atoms of each block to be identified lies in the signal subspace.

To adapt MUSIC to the block-sparse case, we first remark that finding the maxima of p in equation (1) can be replaced by finding the maxima of

$$c(\theta) = \frac{\|P\mathbf{M}\mathbf{a}(\theta)\|}{\|\mathbf{a}(\theta)\|}$$

which is actually the cosine of the angle between the signal subspace and the vector line spanned by $\mathbf{a}(\theta)$. We can generalize this to the block-sparse case, by computing for each j the angle (or its cosine) between an estimation of the signal subspace and the space spanned by the atoms of the sub-dictionary \mathbf{D}_j . This estimated signal subspace is obtained as the space spanned by the first K left singular vectors of the matrix of measurements \mathbf{M} . Note that, as will be shown below, this estimation is not, even in the noiseless case, necessarily of the same dimension as the actual signal subspace.

The cosine of this angle can be easily computed as the largest singular value of $\mathbf{S}^* \mathbf{Q}_j$ where \mathbf{S} and \mathbf{Q}_j are respectively orthogonal bases of the signal subspace and of the span of \mathbf{D}_j [8]. The algorithm is as follows: given a set of measurements in the form of a matrix \mathbf{M} , a set of sub-dictionaries \mathbf{D}_j and a parameter K ,

- compute orthogonal bases \mathbf{Q}_j of the spaces spanned by the sub-dictionaries,
- compute a basis \mathbf{S} of the estimated signal subspace by using the first K left singular vectors of \mathbf{M} ,
- compute the pseudo-spectrum $c(j) = \sigma_1(\mathbf{S}^* \mathbf{Q}_j)$, where σ_1 is the largest singular value of $\mathbf{S}^* \mathbf{Q}_j$.

The support of the signals is estimated by considering the largest S values of the pseudo-spectrum c .

5. THEORETICAL ANALYSIS

We give here necessary conditions for the parameter K of the algorithm, the number of snapshots N_s and the number of measurements N_m to guarantee perfect reconstruction in the case of noiseless measurements. We use here a slightly more abstract formulation of the

model. The signals are in a space E of dimension N_m . The dictionary is composed of L subspaces E_j , with dimension N_c . The space F spanned by the measurements is the sum of S subspaces E'_j , each of them being a R -dimensional subspace of E_j . The algorithm identifies the subspaces E_j as the subspaces such that $E_j \cap F' \neq \{0\}$, where F' is a K -dimensional subspace of F .

To simplify the analysis, we assume that any set of R -dimensional subspaces of S subspaces E_j are in direct sum. This condition is similar to the conditions based on the spark of the dictionary for the standard MMV problem [9], and in particular implies that the number of measurements N_m is larger than SR .

A first glance at the algorithm would indicate that the minimal parameter K , i.e. the rank of the estimated signal subspace F' , should be SR , as it is the dimension of the actual signal subspace. It is however slightly lower:

$$K^{min} = R(S - 1) + 1.$$

Without loss of generality, we can assume that the subspaces to be identified are the first S ones. If $K \leq R(S - 1)$, then F' can be included in $\bigoplus_{j=1}^{S-1} E'_j$ and $F' \cap E_S = \{0\}$. On the other hand, consider the case where F' does not intersect all the subspaces E_j . Without loss of generality, we can assume that F' does not intersect E_S . Then F' is included in $\bigoplus_{j=1}^{S-1} E'_j$, which is of dimension at most $R(S - 1)$, yielding $K \leq R(S - 1)$. Conversely, we have that if $K > R(S - 1)$, any K -dimensional subspace of F intersects all subspaces E_j for $1 \leq j \leq S$.

The number of snapshot should be obviously larger or equal to K : $N_s^{min} = K \geq K^{min}$. The number of measurements also has to be larger or equal to K , but a necessary condition so that only the subspaces E_j intersect F' is that:

$$N_m^{min} = K + N_c.$$

Indeed, for any subspace E_j , $\dim F' \cap E_j = \dim F' + \dim E_j - \dim(F' + E_j) \geq K + N_c - N_m$, which is strictly positive if $N_m < K + N_c$. In this case, F' intersects all the subspaces E_j and the identification is impossible.

When the smallest parameter K possible is chosen, the number of measurements should be larger than

$$N_m^{min} = R(S - 1) + N_c + 1.$$

Note that when $N_c = R = 1$, that is for the standard MUSIC algorithm, all these results reduces to $K = S$, $N_s^{min} = S$ and $N_m^{min} = S + 1$. In the case where $N_c = R$, it is actually possible to recover the signal by forgetting the block structure and using MUSIC. In this case, one should use $K = RS$ and $N_m \geq RS + 1$. When using the block-sparse MUSIC algorithm with $K = RS$, the minimal number of measurements is $N_m^{min} = R(S + 1)$ which is larger than the minimal number of measurements needed for the standard MUSIC algorithm. However, using the minimal value $K = R(S - 1) + 1$ allows to use only $N_m = RS + 1$ measurements.

6. NUMERICAL RESULTS

To support our theoretical analysis, we give results of numerical experiments with random dictionaries. The application to directive source localization is also investigated. Our proposed MUSIC algorithm is compared to a greedy algorithm, Simultaneous Block Orthogonal Matching Pursuit (SBOMP), a combination of Simultaneous Orthogonal Matching Pursuit [3] and Block Orthogonal Matching Pursuit [10].

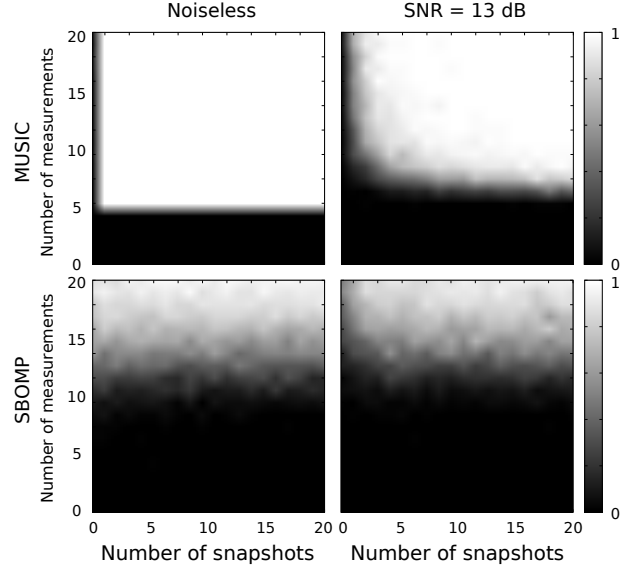


Fig. 1. Probability of recovery with $L = 100$, $R = 1$, $N_c = 4$, $S = 2$, noiseless measurements (left) and noisy measurements, SNR = 13dB (right), random dictionary, (top) MUSIC, (bottom) SBOMP.

6.1. Random dictionaries

We use here random dictionaries and decomposition coefficients, with elements drawn from i.i.d. gaussian probability densities. A dictionary of L subspaces is created from the orthogonalization of $N_m \times N_c$ gaussian matrices. A subset of S matrices is chosen, and the signals are synthesized using random $N_c \times N_s$ matrices of coefficients with rank R .

The subspaces are identified as the S maximal values of the pseudo-spectrum using the minimal value of K . Two sets of parameters are tested, one with rank one matrices (similar to the source localization problem), the other one with full rank matrices (similar to the spectral analysis problem), both with noiseless and noisy measurements (gaussian white noise with SNR = 20 dB). The probability of identification of the subspaces is estimated using 100 trials, with number of snapshots and measurements between 1 and 20. These empirical probabilities are given on figure 1 and 2 for two sets of parameters. For both sets of parameters and noiseless measurements, the probability of recovery with MUSIC is 1 when enough measurements and snapshots (according to the theoretical analysis) are used (2 snapshots and 6 measurements in the first case, 5 snapshots and 7 measurements in the second case). Interestingly, in the full rank case and even when not enough snapshots are available, identification is possible with non-zero probability when using more measurements. The greedy algorithm is able to identify the subspaces with only one snapshot, but needs more measurements. In contrast to the MUSIC algorithm, the transition between probability 0 and probability 1 of recovery is smooth.

Performances with noisy measurements (SNR = 13 dB) are degraded, but not unreasonably so. The MUSIC algorithm performs here slightly better than SBOMP.

6.2. Directive sources

Localization of directive sources is now simulated. For the sake of simplicity, the simulations are done in two dimensions. A linear

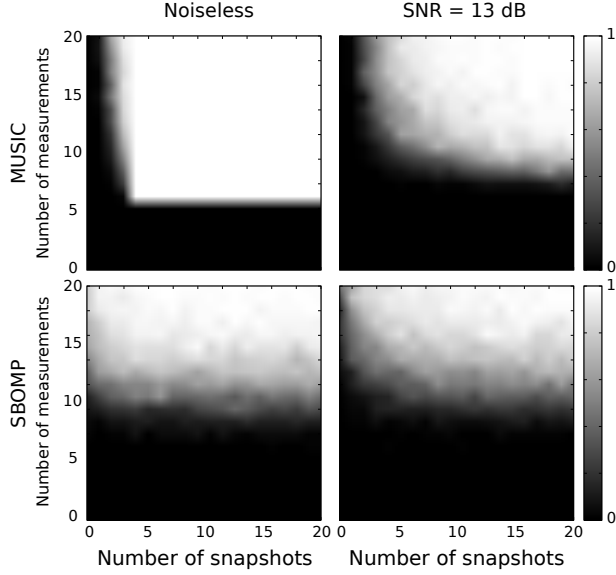


Fig. 2. Probability of recovery with $L = 100$, $R = 2$, $N_c = 2$, $S = 3$, noiseless measurements (left) and noisy measurements, SNR = 13dB (right), random dictionary, (top) MUSIC, (bottom) SBOMP.

antenna of 8 omnidirectional sensors (the minimal number in this case) is used to locate two sources, one fixed, the other moving parallel to the antenna. This antenna is of length 1 m, and the sources are at a distance of 1 m. The SNR is 20 dB, and the wavenumber is $k = 18.5 \text{ m}^{-1}$ corresponding to a frequency of 1 kHz and a wavelength of $\lambda = 34 \text{ cm}$.

The result of the localization of the two sources are given on figure 3 for noiseless measurements and noisy measurements with SNR = 20 dB. In the noiseless case, while the greedy localization methods only gives approximate results, the proposed MUSIC method yields correct estimation of the location of the sources, even when the sources are separated by less than a wavelength λ , indicated on the figure. MUSIC also offers better performances in the noisy case.

7. CONCLUSION

We introduced a variant of the MUSIC algorithm for the case of block-sparse signals. This model, in the case of source localization, is able to take in account directive sources. Our first theoretical and simulation results show that the number of measurements and of snapshots does not only depend on the sparsity and the size of the block, but also on the rank of the coefficient sub-matrix for each block involved in the decomposition. Further works will refine the theoretical analysis, in particular to explain the sharp transition in the rank 1 case and the soft transition in the full rank case of the empirical probability of recovery in function of the number of snapshots, and investigate the connections with recovery of low-rank joint sparse signals [11].

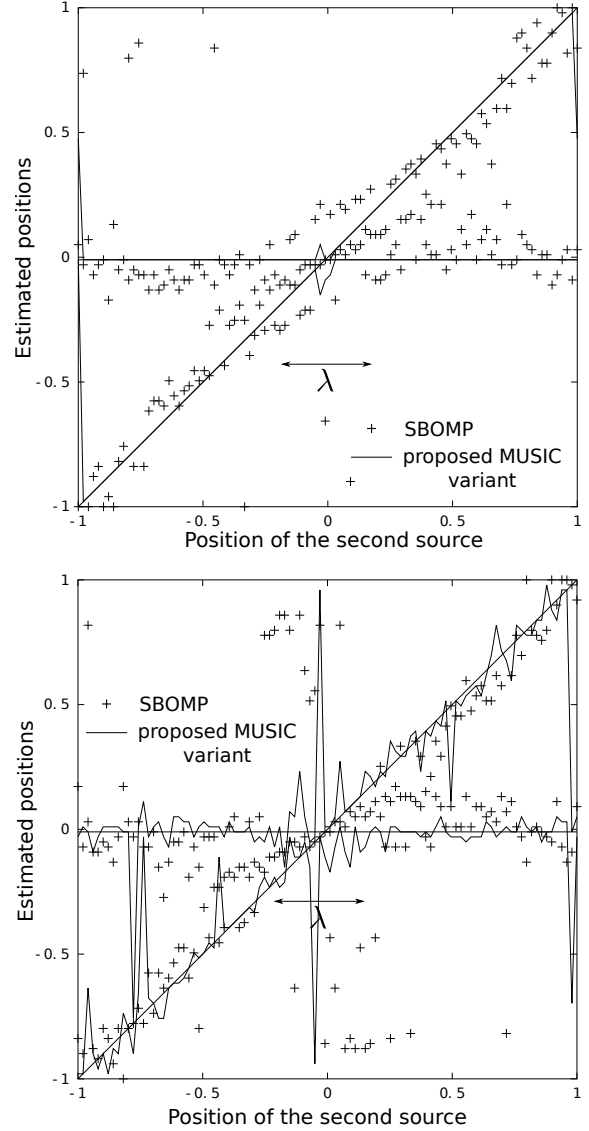


Fig. 3. Estimation of the position of two sources. The first is fixed at position 0, the second moves parallel to the array. Results of the proposed MUSIC variant and the Simultaneous Block Orthogonal Matching Pursuit SBOMP. (top) noiseless measurements, (bottom) SNR = 20 dB.

8. REFERENCES

- [1] K. Lee, Y. Bresler, and M. Junge, "Subspace methods for joint sparse recovery," *Information Theory, IEEE Transactions on*, vol. 58, no. 6, pp. 3613–3641, 2012.
- [2] M.E. Davies and Y.C. Eldar, "Rank awareness in joint sparse recovery," *Information Theory, IEEE Transactions on*, vol. 58, no. 2, pp. 1135–1146, 2012.
- [3] R. Gribonval, H. Rauhut, K. Schnass, and P. Vandergheynst, "Atoms of all channels, unite! average case analysis of multi-channel sparse recovery using greedy algorithms," *Journal of Fourier Analysis and Applications*, vol. 14, no. 5-6, pp. 655–687, 2008.
- [4] R. Schmidt, "Multiple emitter location and signal parameter estimation," *Antennas and Propagation, IEEE Transactions on*, vol. 34, no. 3, pp. 276 – 280, 1986.
- [5] F. Shahbazi Avarvand, A. Ziehe, and G. Nolte, "Music algorithm to localize sources with unknown directivity in acoustic imaging," in *Acoustics, Speech and Signal Processing (ICASSP), 2011 IEEE International Conference on*, 2011, pp. 2744–2747.
- [6] D. Malioutov, M. Cetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Trans. Sig. Proc.*, vol. 53, no. 8, 2005.
- [7] A. Peillot, F. Ollivier, G. Chardon, and L. Daudet, "Acoustic sources joint localization and characterization using compressive sampling," in *Proceedings of Acoustics 2012*, 2012.
- [8] A. Björck and G. H. Golub, "Numerical methods for computing angles between linear subspaces," *Math. Comp.*, vol. 27, pp. 579–594, 1973.
- [9] J. Chen and X. Huo, "Theoretical results on sparse representations of multiple-measurement vectors," *Signal Processing, IEEE Transactions on*, vol. 54, no. 12, pp. 4634–4643, 2006.
- [10] P. Kuppinger Y. C. Eldar and H. Bolcskei, "Block-sparse signals: Uncertainty relations and efficient recovery," *IEEE Trans. Signal Processing*, vol. 58, no. 6, pp. 2636–2646, 2010.
- [11] M. Golbabaee and P. Vandergheynst, "Compressed Sensing of Simultaneous Low-Rank and Joint-Sparse Matrices," submitted to *IEEE Transactions on Information Theory*.